Harmonic structure of one-dimensional quadratic maps

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We study here the ''harmonic structure'' of one-dimensional quadratic maps. The patterns of the structure can be generated with only one initial datum: the symbolic sequence C of the period-1 superstable orbit. All the patterns of the structure are F harmonics (Fourier harmonics). Rules to compose two patterns and rules to calculate F-harmonics are given. The harmonic-structure matrix which contains all the F harmonics in a very compact way by means of the harmonic notation is introduced. [S1063-651X(97)04008-7]

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I. INTRODUCTION

As is well known, all the one-dimensional (1D) quadratic maps are equivalent because they are topologically conjugate [1–3]. This means that any 1D quadratic map can be used to study the others. Therefore we can choose one of them and extend the result to the other ones. For this purpose, we normally use the map $x_{n+1}=x_n^2+c$, which we call the real Mandelbrot map. But, as we showed in a recent work [4], to study 1D quadratic maps through $x_{n+1}=x_n^2+c$ we made use of a subtlety: we use the real axis neighborhood (the antenna) of the Mandelbrot set, which offers graphic advantages. Indeed, the bifurcation diagram has been the normal tool used to analyze the chaotic or periodic behavior of 1D quadratic maps, but not too much can be seen with this tool. If we draw the antenna of the Mandelbrot set with the escape line method [4] much more graphic information can be obtained, because we manage midgets (tiny copies of the Mandelbrot set), cardioids, disks, etc., that we can place and whose periods can be directly measured. However, we must take into account that only the intersection of the Mandelbrot set and the real axis has a sense in the study of the real Mandelbrot map. Therefore it must be clear that when we sporadically talk about a midget (of the Mandelbrot set antenna) we refer to a window (of the $x_{n+1}=x_n^2+c$ map), and when we talk about cardioids or disks (hyperbolic components of the Mandelbrot set antenna, sometimes simply called components) we refer (in the $x_{n+1}=x_n^2+c$ map) to superstable periodic orbits which are born, respectively, from a tangent bifurcation or a pitchfork bifurcation.

The aim of this work is to contribute to the ordering in 1D quadratic maps. We have already published a paper about it [5], where a considerable effort was made to order the superstable periodic orbits. However, that work is only a descriptive and approximate approach to the global ordering. Now, in this work, we accomplish a rigorous treatment of the ordering structure. Indeed, as we shall see later, we use here symbolic sequences instead of periods used in [5], we introduce the harmonic structure which is rigorously calculated, we extend the concept of the harmonic structure to a midget antenna and to a rightward map (like the logistic map), and we introduce the harmonic-structure matrix, a way to represent compactly the symbolic sequences of F-harmonics. All these points, which constitute almost the total of this work, are indeed important contributions with regard to our former work about ordering [5] or others of our works.

As we pointed out before, when we defined a superstable orbit in our former work of Ref. [5], only the period was taken into account; but, that is not enough because several different superstable orbits can have the same period. What rigorously defines each orbit is its symbolic sequence (or pattern) [6–8]. In this work each superstable orbit is always associated to its symbolic sequence. According to Metropolis, Stein, and Stein (MSS) [6] a p-periodic superstable orbit has a symbolic sequence with $p-1$ letters (L's and R's) properly combined. But a pattern corresponding to an orbit of period p with $p-1$ letters can be misleading; and, to avoid that, Zheng and Hao [7] and Schroeder [8] write the symbolic sequence by adding a C at the end [7] or at the beginning [8] of the MSS pattern. We normally use the last procedure to write a symbolic sequence. The meanings of the letters of a pattern are center (C), left (L), and right (R), and they indicate the position of the iterate with regard to the critical point of the map. So, the symbolic sequence (pattern) of the period-3 superstable orbit of the real Mandelbrot map, located at $c=-1.754\,877\,666\ldots$, is CLR.

The importance of Misiurewicz points [9–11] as ''separators'' to order 1D quadratic maps was reported by us [5]. We showed there that the band-merging points are Misiurewicz points, but nothing was said about their symbolic sequences which were introduced later by us in two more recent papers [12,13]. Here, Misiurewicz points are not only reported as separators but their symbolic sequences are rigorously calculated by means of the F harmonics.

The search for order in chaos was early carried out by Sharkovsky [14] (see Sharkovsky *et al.* [15]). Let f_λ: $x\to\lambda x(1-x)$ be the logistic map. Denote by $\lambda[n]$ the least value of the parameter λ for which the map f_λ possesses a cycle with period p. The Sharkovsky theorem ([15], p. 66) says that $\lambda[1]\leqslant\lambda[2]\leqslant\lambda[4]\leqslant\cdots\leqslant\lambda[5\times2]\leqslant\lambda[3\times2]\leqslant\cdots\lambda[5]\leqslant\lambda[3]$.

The Sharkovsky theorem gives a clear ordering of the first appearance superstable periodic orbits (see Fig. 1), but without taking into account either the symbolic sequence of each periodic orbit or the origin of each periodic orbit. On the contrary, the outstanding work of MSS [6] uses both the

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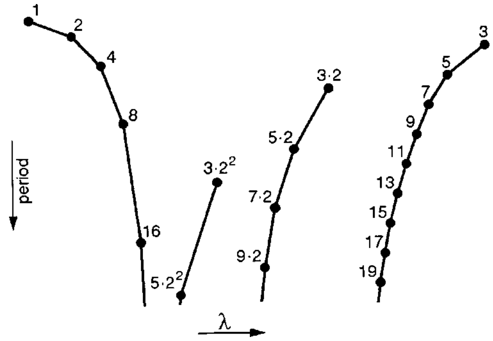

FIG. 1. A sketch of the Sharkovsky theorem for the logistic map $x_{n+1}=\lambda x_n(1-x_n)$. First appearance superstable orbits for periods $p\leqslant 20$ are shown.

symbolic sequence and the pattern generation; however, it is difficult to see any ordering there (see Fig. 2 where we graphically show, as we did in [5], the MSS superstable periodic orbit generation). In the present work we shall make an effort to order superstable orbits in the clearest possible way, at the same time as we show how every superstable orbit is generated and what its symbolic sequence is. However, in the same way as the Sharkovsky ordering only treats a part of the total set of the superstable orbits, the first appearance superstable orbits, we only treat here another part of this set, the last appearance superstable orbits.

In this work we shall obtain what we call ''the harmonic structure'' of a 1D quadratic map which results from the generation of all the ''genes,'' i.e., the superstable orbits of the period-doubling cascade. This harmonic structure is essential for understanding the way the superstable orbits are ordered, and it is constituted by a special type of patterns, that we call the structural patterns. This harmonic structure obtained from the genes is a way of seeing the ordering that shows rigorously the connection between each period-doubling cascade component (gene) and the corresponding chaotic band. Different models of ordering in 1D quadratic maps can be given (we are working in several) and each one has its advantages and disadvantages, but all of them follow the same procedure: filling of the harmonic structure.

We can obtain all the structural patterns by starting out only from the pattern C of the period-1 superstable orbit. Beginning from this pattern C, all the patterns of the period-doubling cascade and the patterns of the last appearance superstable orbits of the chaotic bands are generated. We shall clearly see that the origin of each period-2^n chaotic band $\mathbf{B}_n$ is the disk n of the period-doubling cascade, a disk of period 2^n which is the ''gene'' $\mathbf{G}_n$.

In this article we use the F harmonics (from Fourier) that were introduced by us [13] instead of the MSS-harmonics (from Metropolis, Stein, and Stein). In the same way as the limit of the F harmonics of a cardioid pattern is the tip of the midget that is born in this cardioid [16], we shall see that the limit of the F harmonics of the disk pattern n of the period-doubling cascade whose period is 2^n is the Misiurewicz point m_n that separates the chaotic bands $\mathbf{B}_{n-1}$ and $\mathbf{B}_n$.

As we shall see, there are two types of 1D quadratic maps. We shall introduce two types of composition rules (leftward and rightward rules) in each one. The rules to obtain F harmonics (and F antiharmonics) are simplified cases of the previous rules, and in both cases obey very simple and mnemonic rules.

Finally we shall introduce the harmonic-structure matrix which, in only one formula, shows all the F harmonics of the harmonic structure of a 1D quadratic map. For that, we use the harmonic notation that allows us to put the symbolic sequences of the F harmonics in a very compact way. This new harmonic-structure matrix allows us to see at a glance all the structural patterns; therefore this matrix considerably helps to simplify the always complex vision of the chaos.

II. COMPOSITION RULES

Let us consider the real Mandelbrot map. As we know from [13], a pattern P has even ''L parity'' if it has an even number of L's, and has odd ''L parity'' otherwise. L parity is a concept similar to R parity, introduced by MSS [6] for the logistic map.

After this, let us introduce the composition of two patterns, an augend P_1 and an addend P_2. The pattern P_2 can be added to the pattern P_1 in two different ways: toward the left and toward the right. Let us see the first one.

Definition 1. (Leftward rule) Let P_1 be the pattern of the augend and let P_2 be the pattern of the addend to be added toward the left to P_1. The composition pattern is formed by appending P_2 to P_1 and changing the C of P_2 to L (or R) if the L parity of P_1 is even (or odd).

As a mnemonic rule, we call this composition rule the ''lero'' rule (from L if even and R if odd). Since this addition has direction (leftward in this case) we use the symbol $\overleftarrow{\mp}$. For example, if $P_1=\mathrm{CLR}$ and $P_2=\mathrm{CL}$ then $P_1 \overleftarrow{\mp} P_2=\mathrm{CLR} \overleftarrow{\mp} \mathrm{CL}=\mathrm{CLR}^2\mathrm{L}$, where R^2 means RR. However, for our convenience, we sometimes denote $\mathrm{CLR}\overleftarrow{\mp}\mathrm{CL}$ as $\mathrm{CLR}\overleftarrow{\mathrm{C}}\mathrm{L}$, which is easier to use.

The obtaining of F harmonics of a pattern P that we saw

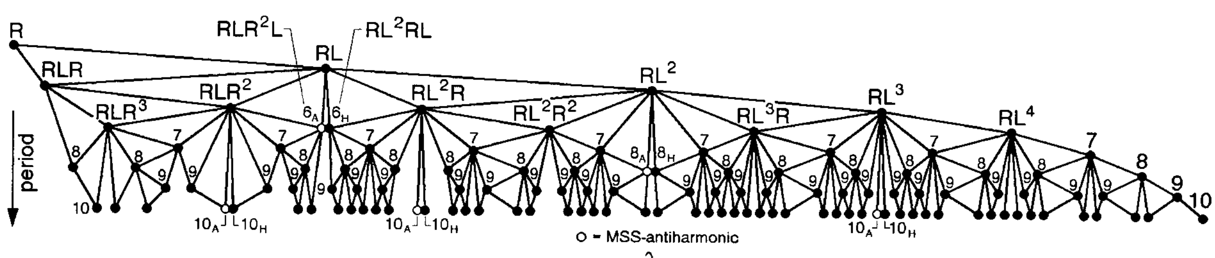

FIG. 2. A sketch of the successive application of the Metropolis, Stein, and Stein theorem in the logistic map $x_{n+1}=\lambda x_n(1-x_n)$ for $p\leqslant 10$. Symbolic sequences for periods $p\leqslant 6$ are shown.

in [13] is a particular case of the previous definition where $P_1 = P_2 = P$, as can be seen as follows.

Definition 2. Let P be a pattern. The first F harmonic of $P, H_{\mathrm{F}}^{(1)}(P)$, is formed by appending P to itself and changing the second C to L (or R) if L parity of P is even (or odd). The second F harmonic of $P, H_{\mathrm{F}}^{(2)}(P)$, is formed by appending P to $H_{\mathrm{F}}^{(1)}(P)$ and changing the second C to L (or R) if L parity of $H_{\mathrm{F}}^{(1)}(P)$ is even (or odd), and so on.

In this definition, we have changed the Schroeder rule [8] for the calculation of the patterns of MSS harmonics in order to calculate the patterns of F harmonics [13]. As we shall see afterwards, we generate all the structural patterns by the only use of this rule of F harmonics.

Let us see now the second type of composition, the addition toward the right.

Definition 3. (Rightward rule.) Let P_1 be the pattern of the augend and let P_2 be the pattern of the addend to be added toward the right to P_1. The composition pattern is formed by appending P_2 to P_1 and changing the C of P_2 to R (or L) if the L parity of P_1 is even (or odd).

As a mnemonic rule, we call this composition rule the ''relo'' rule (from R if even and L if odd). Since this addition has direction (rightward in this case) we use the symbol $\overrightarrow{+}$. For example, if $P_1 = \mathrm{CLR}$ and $P_2 = \mathrm{CL}$ then $P_1 \overrightarrow{+} P_2 = \mathrm{CLR} \overrightarrow{+} \mathrm{CL} = \mathrm{CLRL}^2$, where L^2 means LL. However, for our convenience, we sometimes denote $\mathrm{CLR} \overrightarrow{+} \mathrm{CL}$ as $\mathrm{CLR}\vec{\mathrm{C}}\mathrm{L}$, which is easier to use.

The obtaining of F antiharmonics of a pattern P that we saw in [13] is a particular case of the previous definition where $P_1 = P_2 = P$, as can be seen as follows.

Definition 4. Let P be a pattern. The first F antiharmonic of $P, A_{\mathrm{F}}^{(1)}(P)$, is formed by appending P to itself and changing the second C to R (or L) if L parity of P is even (or odd). The second F antiharmonic of $P, A_{\mathrm{F}}^{(2)}(P)$, is formed by appending P to $A_{\mathrm{F}}^{(1)}(P)$ and changing the second C to R (or L) if L parity of $A_{\mathrm{F}}^{(1)}(P)$ is even (or odd), and so on.

Just like a MSS antiharmonic is a purely formal construction [6] and never corresponds to a periodic orbit, an F antiharmonic is also a purely formal construction and never corresponds to a periodic orbit either. But, although they have no real existence, they act to generate new patterns when we want to fill up the harmonic structure, as we plan to show in works we are preparing about different models of pattern generation. However, in the case of the structural patterns that we are treating here only F harmonics are present.

By speaking according to the Mandelbrot set terminology, we have two types of structural components: disks and cardioids. In the following figures a disk is depicted as a full circle, and a cardioid as a circle with a full half part and an empty half part. As can be seen in all the figures, the first F harmonic of a component is a disk (belonging to the period-doubling cascade and is generated by means of a pitchfork bifurcation), all the other F harmonics are cardioids (belonging to the chaotic region and are generated by means of tangent bifurcations), and the limit of the F harmonics of a component is a Misiurewicz point.

III. THE HARMONIC STRUCTURE OF 1D QUADRATIC MAPS

Let us see how to generate the chaotic bands in the real Mandelbrot map by beginning just at the origin, i.e., at the period-1 superstable orbit of symbolic sequence C. For this purpose, we shall be assisted by Fig. 3, where in the upper parts we depict a sketch of the F harmonics where periods and symbolic sequences are shown and in the lower parts we depict the corresponding Mandelbrot set antenna zones by means of the escape lines method [4]. In (a) we show the F harmonics of C obtained in accordance with the lero rule. To form the first F harmonic of C we add a C to the C, i.e., we write CC and we change the second C into a L, since the L parity of C is even. Therefore the first F harmonic of C is $H_{\mathrm{F}}^{(1)}(\mathrm{C}) = \mathrm{C} \overleftarrow{+} \mathrm{C} = \mathrm{C}\overleftarrow{\mathrm{C}} = \mathrm{CL}$. To form the second F harmonic of C we add a C to the first one, i.e., we write CLC and we change the second C into a R, since the L parity of CL is odd, and we obtain $H_{\mathrm{F}}^{(2)}(\mathrm{C}) = \mathrm{CLR}$.

By applying the same procedure, we obtain that the third and fourth F harmonics of C are $H_{\mathrm{F}}^{(3)}(\mathrm{C}) = \mathrm{CLR}^2$ and $H_{\mathrm{F}}^{(4)}(\mathrm{C}) = \mathrm{CLR}^3$. As is well known [8], $\mathrm{CLR}, \mathrm{CLR}^2, \mathrm{CLR}^3, \ldots$ $(\mathrm{CRL}, \mathrm{CRL}^2, \mathrm{CRL}^3, \ldots,$ in the logistic map) are the symbolic sequences of the last appearance superstable orbits (although in the cases of CL and CLR they are also first appearance superstable orbits since there is only one superstable orbit of period 2 and only one of period 3). As we already showed [16], the limit of the F harmonics of a pattern can be calculated. In this case the limit $H_{\mathrm{F}}^{(\infty)}(\mathrm{C})$ is the Misiurewicz point with preperiod 2 and period 1, $m_0 = M_{2,1}$, whose symbolic sequence is (CL)R. In the Mandelbrot set this point is the main antenna end, tip(C) [16], whose parameter value is $c = -2$. In the real Mandelbrot map this point is the boundary crisis point (according to Grebogi *et al.* [17]). Therefore the F harmonic development of C goes over the whole antenna, from the beginning to the end.

We started from the period-1 superstable orbit C placed in the periodic region or Feigenbaum region. The first F harmonic of C is the period-2 superstable orbit of the period-doubling cascade. All the other F harmonics of C are superstable orbits placed in the period-1 (2^0) chaotic band $\mathbf{B}_0$ and are the last appearance superstable orbits of this band. If we consider the period-2^0 superstable orbit C as a ''gene'' $\mathbf{G}_0$, then the F harmonics of the gene $\mathbf{G}_0$ generate the period-2^0 chaotic band $\mathbf{B}_0$. However, a period-2^1 superstable orbit placed in the periodic region is also generated; we shall call this the ''bridge pattern.'' Let us see what happens when this orbit is used as a new gene $\mathbf{G}_1$.

Let us see now Fig. 3(b) where we show the harmonics of $\mathbf{G}_1 = \mathrm{CL}$ (the former bridge pattern). To form its F harmonics we add CL to the previous one and we change the second C into a L or a R, in accordance with the lero rule. So, we obtain that the first, second, third, and fourth F harmonics of $\mathbf{G}_1$ are $H_{\mathrm{F}}^{(1)}(\mathbf{G}_1) = \mathrm{CLRL}$, $H_{\mathrm{F}}^{(2)}(\mathbf{G}_1) = \mathrm{CLRL}^3$, $H_{\mathrm{F}}^{(3)}(\mathbf{G}_1) = \mathrm{CLRL}^5$, and $H_{\mathrm{F}}^{(4)}(\mathbf{G}_1) = \mathrm{CLRL}^7$. The limit of these harmonics is the Misiurewicz point $m_1 = M_{3,1} = (\mathrm{CLR})\mathrm{L}$, placed in $c = -1.543\,689\,012\ldots$ [12,13], that separates the period-1 chaotic band $\mathbf{B}_0$ and the period-2 chaotic band $\mathbf{B}_1$. The first harmonic generated from CL is the pattern CLRL of period 4 (2^2), which corresponds to the second superstable orbit of the period-doubling cascade of the periodic region of C, which we called a bridge pattern, and all the other F harmonics of CL are superstable orbits placed in the period-2 (2^1) chaotic band $\mathbf{B}_1$, and are the last appearance superstable or-

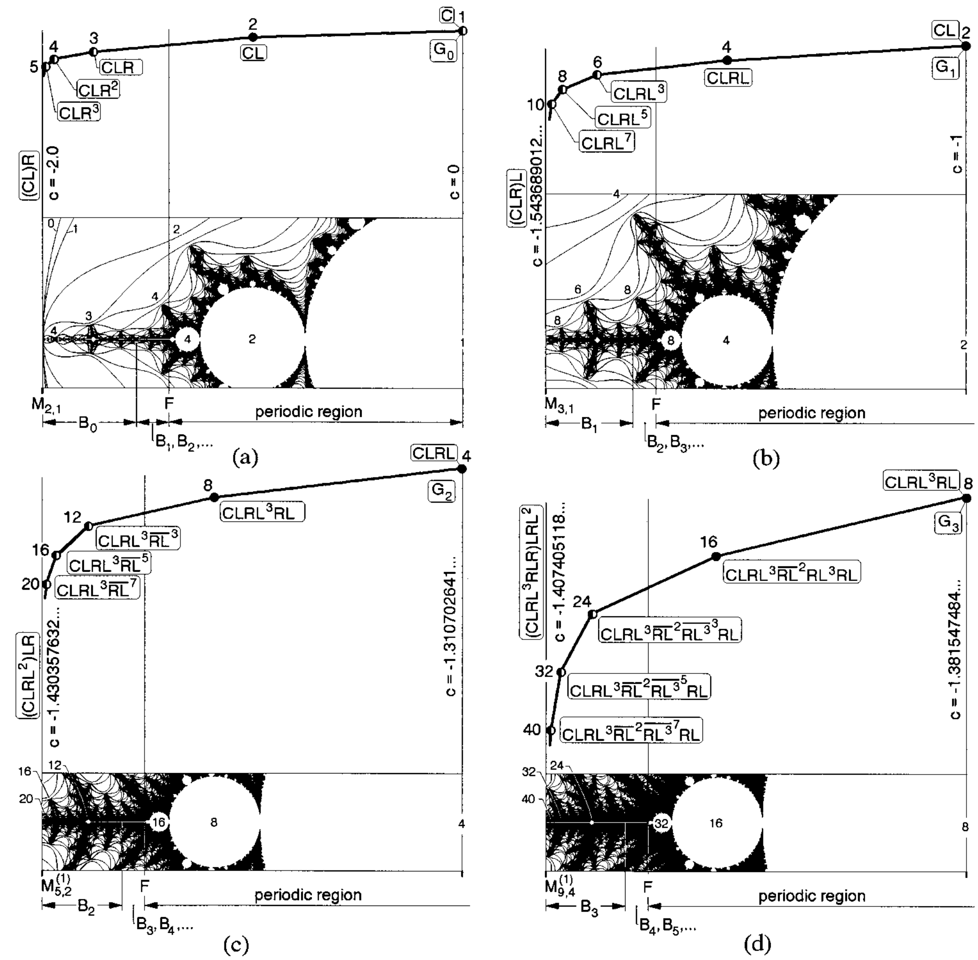

FIG. 3. A sketch of the F harmonics of the first four superstable orbits of the period-doubling cascade in the real Mandelbrot map. The harmonic generation of last appearance superstable patterns of chaotic bands is shown. (a) Period-1 chaotic band $\mathbf{B}_0$; (b) period-2 chaotic band $\mathbf{B}_1$; (c) period-4 chaotic band $\mathbf{B}_2$; (d) period-8 chaotic band $\mathbf{B}_3$.

bits of this band. Again, if we consider CL as a gene $\mathbf{G}_1$, we have that the F harmonics of the gene $\mathbf{G}_1$ generate the period-2^1 chaotic band $\mathbf{B}_1$ (and a bridge pattern that will be the new gene $\mathbf{G}_2=$CLRL).

Let us see now Fig. 3(c) where we show the F harmonics of the previous bridge pattern, the gene $\mathbf{G}_2=$CLRL. The first, second, third, and fourth harmonics of $\mathbf{G}_2$ are $H_{\mathrm{F}}^{(1)}(\mathbf{G}_2)=\mathrm{CLRL}^3\mathrm{RL}$, $H_{\mathrm{F}}^{(2)}(\mathbf{G}_2)=\mathrm{CLRL}^3\overline{\mathrm{RL}}^3$, $H_{\mathrm{F}}^{(3)}(\mathbf{G}_2)=\mathrm{CLRL}^3\overline{\mathrm{RL}}^5$, and $H_{\mathrm{F}}^{(4)}(\mathbf{G}_2)=\mathrm{CLRL}^3\overline{\mathrm{RL}}^7$, the first a bridge pattern in the periodic region and the others superstable orbits placed in the period-2^2 chaotic band $\mathbf{B}_2$. (We write n times RL as $\overline{\mathrm{RL}}^n$, because we only use brackets in Misiurewicz point preperiods.) The limit of these harmonics is the Misiurewicz point $m_2=M_{5,2}^{(1)}=(\mathrm{CLRL}^2)\mathrm{LR}$ [12,13], that separates the period-2 chaotic band $\mathbf{B}_1$ and the period-4 chaotic band $\mathbf{B}_2$. Therefore the F harmonics of the gene $\mathbf{G}_2$ generate the period-2^2 chaotic band $\mathbf{B}_2$, and a new bridge pattern $\mathbf{G}_3=\mathrm{CLRL}^3\mathrm{RL}$ that is the gene of the next chaotic band.

Finally, let us see Fig. 1(d), where we show the F harmonics of the previous bridge pattern of period 8 (2^3), that is, the gene $\mathbf{G}_3=\mathrm{CLRL}^3\mathrm{RL}$. A new bridge pattern of period 16 in the periodic region, that is, the gene $\mathbf{G}_4$, and the last appearance superstable orbits of the period-8 chaotic band $\mathbf{B}_3$ are generated. The limit of these superstable orbits is the Misiurewicz point $m_3=M_{9,4}^{(1)}$ [12] that separates the period-4 chaotic band $\mathbf{B}_2$ and the period-8 chaotic band $\mathbf{B}_3$.

Generalizing, the F harmonics of the disk n of the period-doubling cascade, the gene $\mathbf{G}_n$, generates the last appearance cardioids of the period-2^n chaotic band $\mathbf{B}_n$, and a bridge disk of the period-doubling cascade, the gene $\mathbf{G}_{n+1}$. Likewise, $H_{\mathrm{F}}^{(\infty)}(\mathbf{G}_n)$ is a Misiurewicz point $m_n=M_{2^i+1,2^{i-1}}$ [5], a primary separator (or band-merging point) of the chaotic bands $\mathbf{B}_{n-1}$ and $\mathbf{B}_n$.

This double procedure (disks and chaotic band generation) continues indefinitely, and both meet in the Feigenbaum point [18], periodic disks on the right and chaotic bands on the left.

Every pattern of the period-doubling cascade is the gene of the corresponding chaotic band. The set of the F harmonics of all the genes is what we call ''the harmonic structure'' and is schematically shown in Fig. 4. The patterns of the harmonic structure are called structural patterns and all of them are F harmonics. $H_{\mathrm{F}}^{(1)}(\mathbf{G}_n)$ is always a disk, $H_{\mathrm{F}}^{(m)}(\mathbf{G}_n)$ where $1<m<\infty$ is a cardioid, and $H_{\mathrm{F}}^{(\infty)}(\mathbf{G}_n)$ is a Misiurewicz point.

What we call here ''the harmonic structure'' was in part

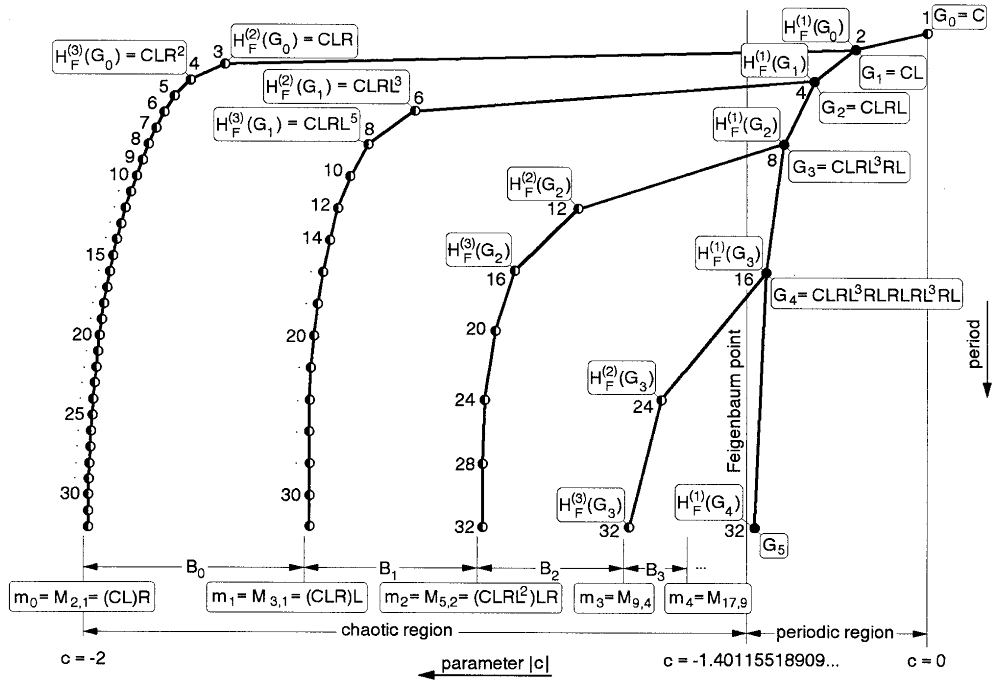

FIG. 4. A sketch of the harmonic structure of the real Mandelbrot map. The F harmonics of each period-doubling cascade superstable orbit, the gene $\mathbf{G}_n$, generate the corresponding chaotic band $\mathbf{B}_n$ and the Misiurewicz point m_n that separates the chaotic bands $\mathbf{B}_{n-1}$ and $\mathbf{B}_n$.

already seen in a previous work [5], but it is here where the harmonic structure has rigorously been calculated. In Fig. 4 we can see the periodic region and the chaotic region separated by the Feigenbaum point. Likewise, the chaotic region is divided in an infinity of chaotic bands $\mathbf{B}_n$, separated by Misiurewicz points called separators, m_n, $0 \leqslant n \leqslant \infty$. Each structural pattern and each separator is determined by starting with the only datum of the pattern C, and by applying the leftward rule to the successive genes. Structural patterns are placed in the figure from $c=0$ to $c=-2$ according to the value of their parameter, and the greater the period of the superstable orbit the lower the position.

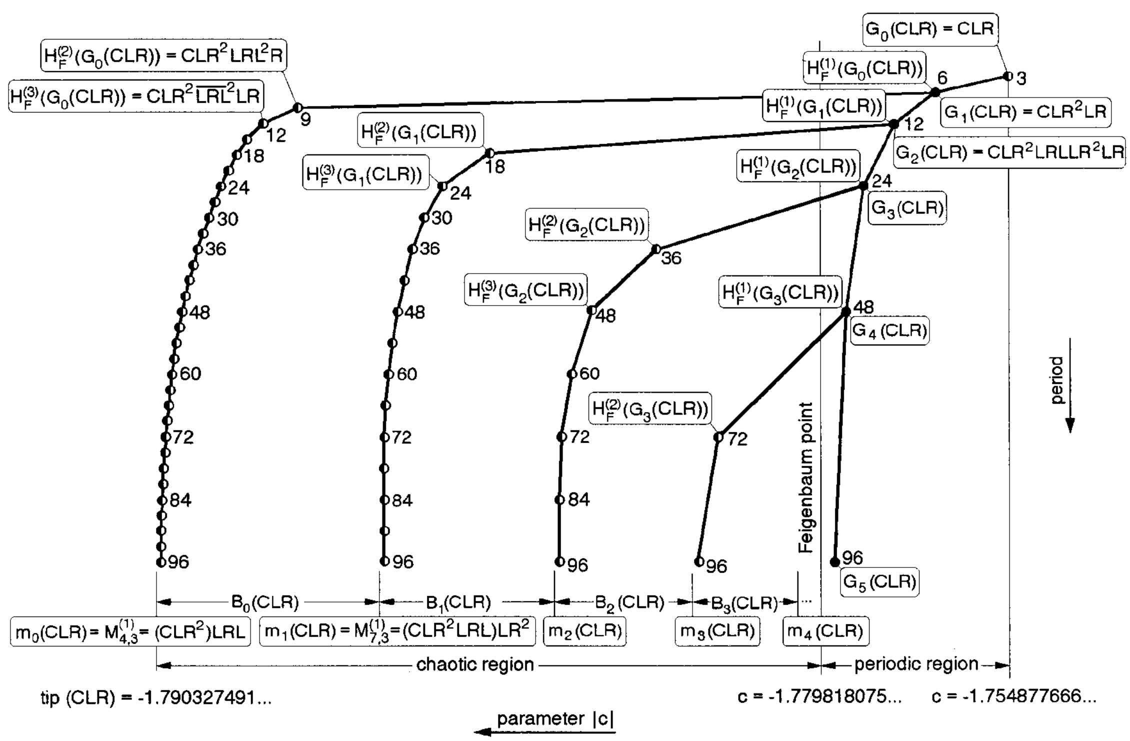

FIG. 5. A sketch of the harmonic structure of the antenna of the midget CLR of the Mandelbrot set antenna. This harmonic structure is equivalent to the harmonic structure of the whole Mandelbrot set antenna.

TABLE I. Parity and rules for each type of 1D quadratic map.

Type of 1D quadratic map	Leftward map	Rightward map
Parity	L parity	R parity
Rule for F harmonics	leftward (lero) rule	rightward (relo) rule
Rule for F antiharmonics	rightward (relo) rule	leftward (lero) rule

IV. THE HARMONIC STRUCTURE OF THE ANTENNA OF A MIDGET

Let us again use the Mandelbrot set terminology, because here it is easier to speak about midgets than about windows. In the Mandelbrot set antenna, every midget is a tiny copy of the whole Mandelbrot set. Therefore the antenna of a midget is a tiny copy of the antenna of the Mandelbrot set (obviously we suppose we are always in the real part of the Mandelbrot set antenna). Hence every antenna of a midget should have the same harmonic structure as the harmonic structure of the antenna of the whole Mandelbrot set.

In fact, let us consider, for example, the midget antenna that is born in the period-3 cardioid CLR (see Fig. 5, where the harmonic structure of the midget CLR is shown). The cardioid CLR can be considered the gene $\mathbf{G}_0(\mathrm{CLR})$. The first, second, third, etc., F harmonics of CLR are $H_{\mathrm{F}}^{(1)}(\mathbf{G}_0(\mathrm{CLR}))=\mathrm{CLR}^2\mathrm{LR}$, $H_{\mathrm{F}}^{(2)}(\mathbf{G}_0(\mathrm{CLR}))=\mathrm{CLR}^2\mathrm{LRL}^2\mathrm{R}$, $H_{\mathrm{F}}^{(3)}(\mathbf{G}_0(\mathrm{CLR}))=\mathrm{CLR}^2\overline{\mathrm{LRL}}^2\mathrm{LR},\ldots,$ which finish in the Misiurewicz point $m_0(\mathrm{CLR})=H_{\mathrm{F}}^{(\infty)}(\mathbf{G}_0(\mathrm{CLR}))=M_{4,3}^{(1)}=(\mathrm{CLR}^2)\mathrm{LRL}$ that is the end of the CLR midget antenna, tip(CLR), placed in $c=-1.790\,327\,491\ldots$ [16]. The first F harmonic of CLR is a bridge pattern in the periodic region of the midget, and all the others are the last appearance cardioids of the period-3 (3×2^0) chaotic band $\mathbf{B}_0(\mathrm{CLR})$. Therefore the F harmonics of the gene $\mathbf{G}_0(\mathrm{CLR})$ generate the partial harmonic structure of the chaotic band $\mathbf{B}_0(\mathrm{CLR})$ and the tip of the midget CLR, tip(CLR). Starting from the bridge disk $\mathbf{G}_1(\mathrm{CLR})=\mathrm{CLR}^2\mathrm{LR}$ of period 6, the last appearance cardioids of the period-6 (3×2^1) chaotic band $\mathbf{B}_1(\mathrm{CLR})$ are generated, and a new bridge pattern of period 12 that will be the gene $\mathbf{G}_2(\mathrm{CLR})$ of the period-12 (3×2^2) chaotic band $\mathbf{B}_2(\mathrm{CLR})$. Likewise, $H_{\mathrm{F}}^{(\infty)}(\mathbf{G}_1(\mathrm{CLR}))=m_1(\mathrm{CLR})=M_{7,3}^{(1)}=(\mathrm{CLR}^2\mathrm{LRL})\mathrm{LR}^2$ that is a Misiurewicz point that separates the chaotic bands $\mathbf{B}_0(\mathrm{CLR})$ and $\mathbf{B}_1(\mathrm{CLR})$. And so on.

In general, the F harmonics of the disk n of the period-doubling cascade of CLR, the gene $\mathbf{G}_n(\mathrm{CLR})$, generate the last appearance components of the chaotic band $\mathbf{B}_n(\mathrm{CLR})$, and also a new gene $\mathbf{G}_{n+1}(\mathrm{CLR})$. Besides, $H_{\mathrm{F}}^{(\infty)}(\mathbf{G}_n(\mathrm{CLR}))=m_n(\mathrm{CLR})$ is the Misiurewicz point that separates the chaotic bands $\mathbf{B}_{n-1}(\mathrm{CLR})$ and $\mathbf{B}_n(\mathrm{CLR})$. The set of the F harmonics of all the genes of CLR is the harmonic structure of CLR; and, as can clearly be seen in Fig. 5, it is equivalent to the harmonic structure of the whole Mandelbrot set antenna.

V. THE HARMONIC STRUCTURE OF ANY TYPE OF 1D QUADRATIC MAP

There are two types of 1D quadratic maps: leftward maps and rightward maps. In the case of a leftward map, when we go through the period-doubling cascade orbits from low periods to great periods, we move towards the left. However, in

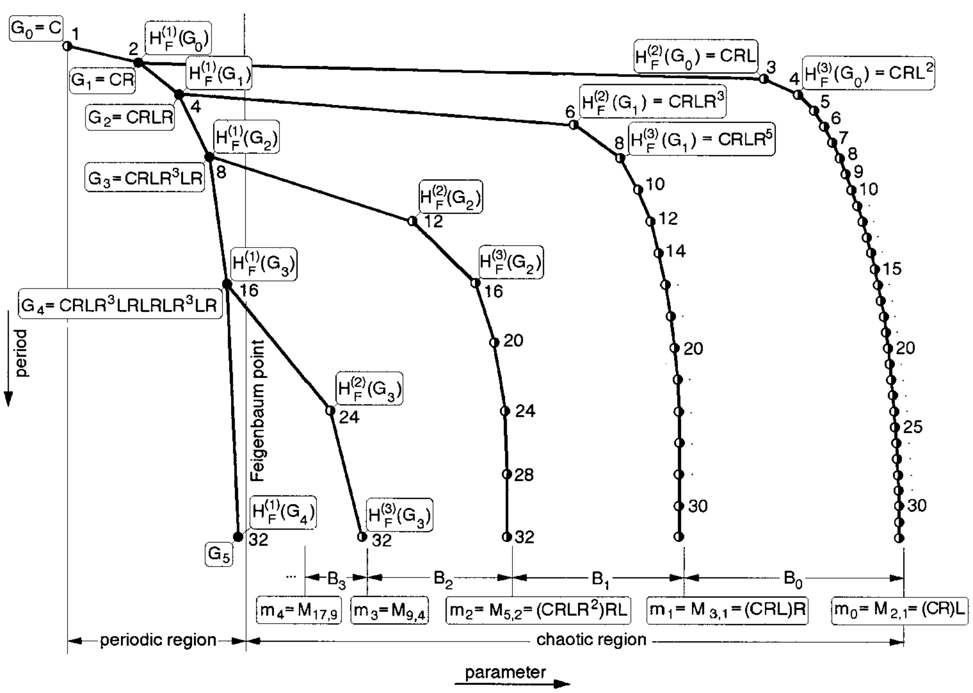

FIG. 6. A sketch of the harmonic structure of a rightward 1D quadratic map.

the case of a rightward map, when we go through the period-doubling cascade orbits from low periods to great periods, we move towards the right. Within the same type of 1D quadratic maps, the same superstable orbits have the same symbolic sequences. In the case of different types of 1D quadratic maps, the symbolic sequence of an orbit is obtained from the symbolic sequence of the same orbit of the other type by interchanging the L's and R's. We can see that in MSS [6] and in our works L's and R's are interchanged because MSS use the logistic map, a rightward map, and we use the real Mandelbrot map, a leftward map.

Hence, if we wanted to obtain the harmonic structure of a rightward map, as the logistic map, definitions 1–4 that were given for a leftward map have to be changed. In accordance with all we saw before, in leftward maps the parity to be used is the L parity and the rule used to calculate the F harmonics is the leftward (lero) rule. Likewise, in rightward maps the parity to be used is the R parity and the rule used to calculate the F harmonics is the rightward (relo) rule. Therefore F harmonics are always bound to the same letter. So, leftward F harmonics are bound to the L: leftward maps, L parity, and leftward (lero) rule; and rightward, F harmonics are bound to the R: rightward maps, R parity, and rightward (relo) rule. The structural patterns are F harmonics, and we can go from the origin (the first periodic orbit with the symbolic sequence C) to each structural pattern only through F harmonics. This means that in every step we always have the same direction as the direction of the type of map we are working with. That is not the case of F antiharmonics where the direction of the map and the rule are not the same. All this is shown in Table I, where we show the parity and the rule that have to be applied to calculate F harmonics (and F antiharmonics) for each type of 1D quadratic map.

In Fig. 6 we have depicted the harmonic structure of a rightward map. Indeed, when we go through the period-doubling cascade orbits from low periods to great periods, we move towards the right, against what happened in leftward maps, where we moved towards the left. Likewise, the corresponding structural patterns have interchanged their L's and R's with regard to leftward maps.

VI. THE HARMONIC STRUCTURE MATRIX

We can give a matrix, the harmonic-structure matrix, to compactly represent all the F harmonics of the harmonic structure. For that, we use what we call the harmonic notation that allows us to put in a very compact way the symbolic sequences of the F harmonics. This matrix is

$$(H_{\mathrm{F}}^{(j)}(\mathbf{G}_i))=\begin{pmatrix} \mathbf{G}_0 & H_{\mathrm{F}}^{(1)}(\mathbf{G}_0) & H_{\mathrm{F}}^{(2)}(\mathbf{G}_0) & \dots & H_{\mathrm{F}}^{(j)}(\mathbf{G}_0) & \dots & m_0 \\ \mathbf{G}_1 & H_{\mathrm{F}}^{(1)}(\mathbf{G}_1) & H_{\mathrm{F}}^{(2)}(\mathbf{G}_1) & \dots & H_{\mathrm{F}}^{(j)}(\mathbf{G}_1) & \dots & m_1 \\ \mathbf{G}_2 & H_{\mathrm{F}}^{(1)}(\mathbf{G}_2) & H_{\mathrm{F}}^{(2)}(\mathbf{G}_2) & \dots & H_{\mathrm{F}}^{(j)}(\mathbf{G}_2) & \dots & m_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{G}_i & H_{\mathrm{F}}^{(1)}(\mathbf{G}_i) & H_{\mathrm{F}}^{(2)}(\mathbf{G}_i) & \dots & H_{\mathrm{F}}^{(j)}(\mathbf{G}_i) & \dots & m_i \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{G}_\infty & H_{\mathrm{F}}^{(1)}(\mathbf{G}_\infty) & H_{\mathrm{F}}^{(2)}(\mathbf{G}_\infty) & \dots & H_{\mathrm{F}}^{(j)}(\mathbf{G}_\infty) & \dots & m_\infty \end{pmatrix},$$

a matrix $i\times j$ where $0\leqslant i\leqslant\infty$ and $0\leqslant j\leqslant\infty$. The first column $[j=0$, because $\mathbf{G}_i=H_{\mathrm{F}}^{(0)}(\mathbf{G}_i)]$ is the gene column which is constituted by the superstable orbits of the period-doubling cascade that finishes in the Feigenbaum point $\mathbf{G}_\infty=F$. The second column, the first F harmonics of a gene, has the following property: $H_{\mathrm{F}}^{(1)}(\mathbf{G}_i)=\mathbf{G}_{i+1}$, i.e., the first F harmonic of a gene is the next gene. The last column $(j=\infty)$ is the band-merging points column m_i because it is constituted by the Misiurewicz points that merge (or separate) two successive chaotic bands. This band-merging points column also finishes in the Feigenbaum point $m_\infty=\mathrm{F}$. Since the first and the last elements of the last row finish in the Feigenbaum point F, all the elements of this row have to be F.

Each row i has a first element, for $j=0$, $\mathbf{G}_i=H_{\mathrm{F}}^{(0)}(\mathbf{G}_{\mathrm{i}})$, which is the gene of this row, a second element, for $j=1$, which is the gene of the next row, and all the other elements, for $1<j<\infty$, which are the last appearance superstable orbits of the chaotic band $\mathbf{B}_{\mathrm{i}}$. The last element, for $j=\infty$, is the band-merging point m_i of the bands $\mathbf{B}_{i-1}$ and $\mathbf{B}_i$.

The general term $H_{\mathrm{F}}^{(j)}(\mathbf{G}_{\mathrm{i}})$ is the jth F harmonic of the gene $\mathbf{G}_{\mathrm{i}}$, and can easily be calculated by applying the corresponding rule according to the type of 1D quadratic map. The number of letters of $\mathbf{G}_i$ is $k\times 2^i$, where k is the number of letters of $\mathbf{G}_0$. Therefore the number of letters of the pattern $H_{\mathrm{F}}^{(j)}(\mathbf{G}_i)$ is $(j+1)k\times 2^i$.

For example, in the case of the real Mandelbrot map, the pattern of the second last appearance superstable orbit of the chaotic band $\mathbf{B}_1$ of the window CLR $(i=1$, $j=3$, $k=3)$ has 24 letters. The harmonic notation of the orbit is $H_{\mathrm{F}}^{(3)}(\mathbf{G}_1(\mathrm{CLR}))$, whose symbolic sequence is $\mathrm{CLR^2LRL^2R^2}\overline{\mathrm{LR^2}}^4\mathrm{LR}$. The parameter value, $c=-1.781\,870\,007\,610\,51\ldots$, is obtained by means of the method given by Kaplan [19].

VII. CONCLUSIONS

The real axis neighborhood of the Mandelbrot set is used to study the harmonic structure of a 1D quadratic map. We base this study in the F harmonics [13] instead of MSS harmonics [6], which are clearly different.

We introduce composition rules for the addition of patterns (towards the left and towards the right), and rules to obtain F harmonics (and F antiharmonics) which are simplifications of the previous ones.

The F harmonics of the nth superstable orbit of the period-doubling cascade (the gene $\mathbf{G}_n$ whose period is 2^n) generate the $(n+1)$th superstable orbit of the period-doubling cascade (the gene $\mathbf{G}_{n+1}$ whose period is 2^{n+1}) and the last appearance superstable orbits of the period-2^n chaotic band $\mathbf{B}_n$. The limit of F harmonics of $\mathbf{G}_n$ is the Misiurewicz point m_n that separates the period-2^{n-1} chaotic band $\mathbf{B}_{n-1}$ and the period-2^n chaotic band $\mathbf{B}_n$. Therefore the set of F harmonics of all the superstable orbits of the period-doubling cascade gives us the harmonic structure of the 1D quadratic map. The harmonic structure has the period-doubling cascade separated from the chaotic region by the Feigenbaum point, and the chaotic bands separated by Misiurewicz points.

The patterns of the harmonic structure are the structural patterns. All the structural patterns can be obtained starting from only one datum: the period-1 superstable orbit of symbolic sequence C.

The harmonic structure of a midget has also been obtained. The harmonic structure of each midget is similar to the harmonic structure of the period-1 superstable orbit of symbolic sequence C.

For all the generation process of new structural patterns we use simple mnemonic rules. There are two types of 1D quadratic maps: leftward maps and rightward maps. To calculate the F harmonics of leftward maps we use the L parity and the leftward rule (lero rule), and to calculate the F harmonics of rightward maps we use the R parity and the rightward rule (relo rule).

The harmonic-structure matrix which contains all the F harmonics in only one formula has been introduced. The harmonic notation that allows us to write the symbolic sequences of the F harmonics in a very compact way has been used in the harmonic-structure matrix.

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